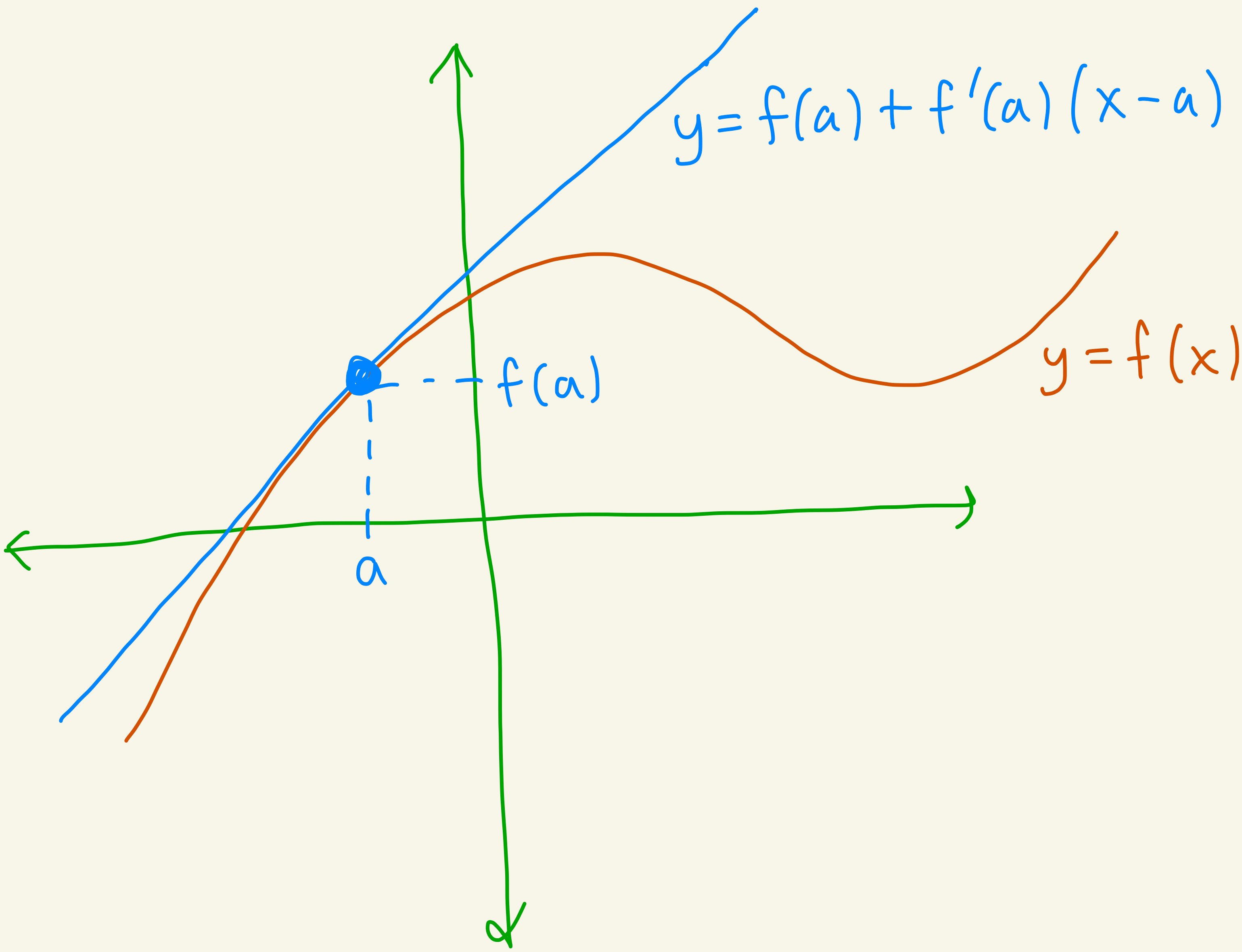
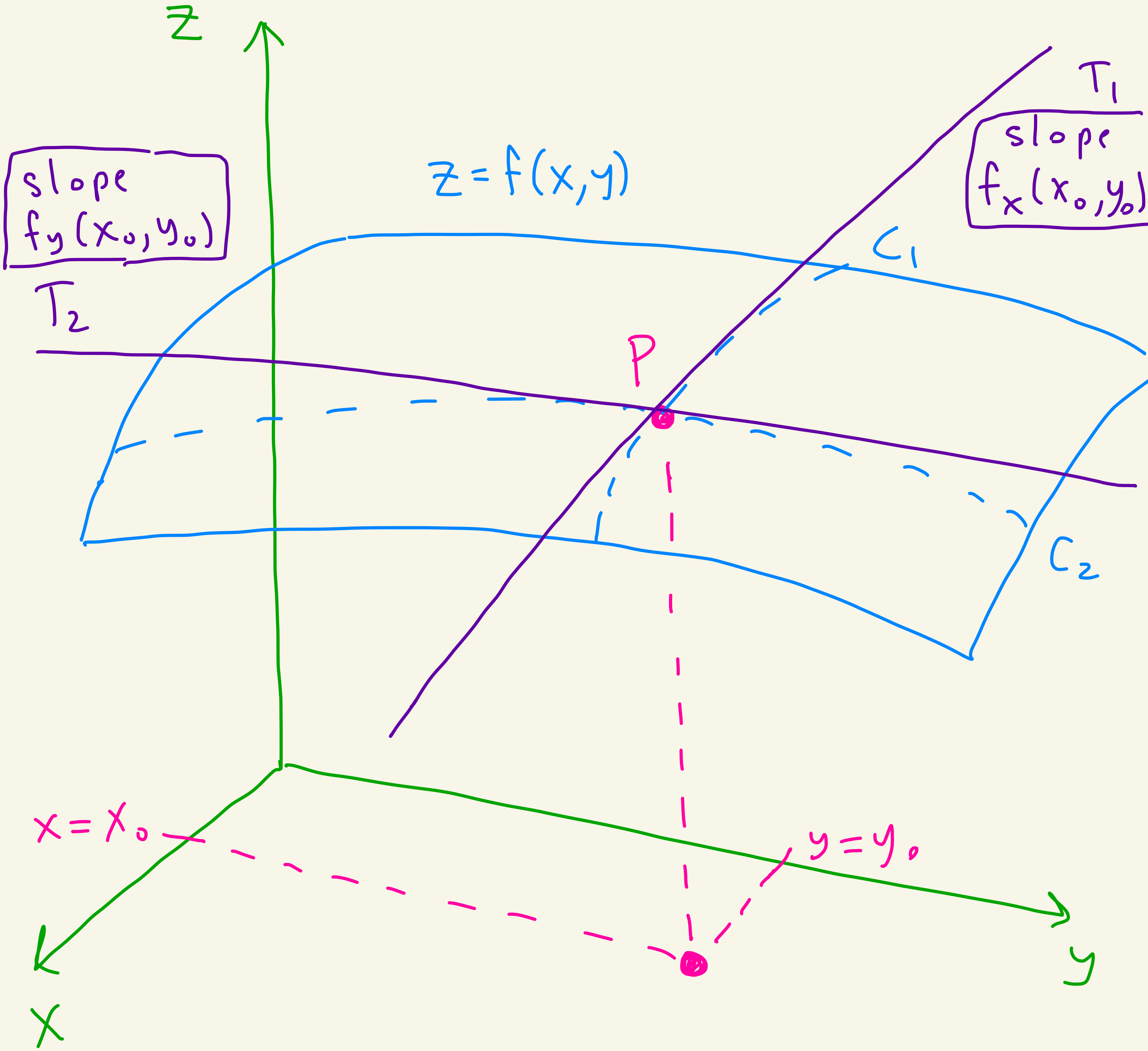
Topic 4 – Tangent planes and local minimum and maximums

Recall the tangent line from Calc I.



Let's generalize to surfaces.
Suppose that a surface S is given by $z = f(x,y)$ where f has continuous first partial derivatives f_x and f_y.
Let $P = (x_0, y_0, z_0)$ be a point on S.
Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $x = x_0$ and $y = y_0$ with S. Then P lies at the intersection of C_1 and C_2.
Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. We know that T_1 has slope $f_x(x_0, y_0)$ and T_2 has slope $f_y(x_0, y_0)$



Let T be the plane that contains T_1 and T_2.

T has an equation of the form

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Dividing by c we get

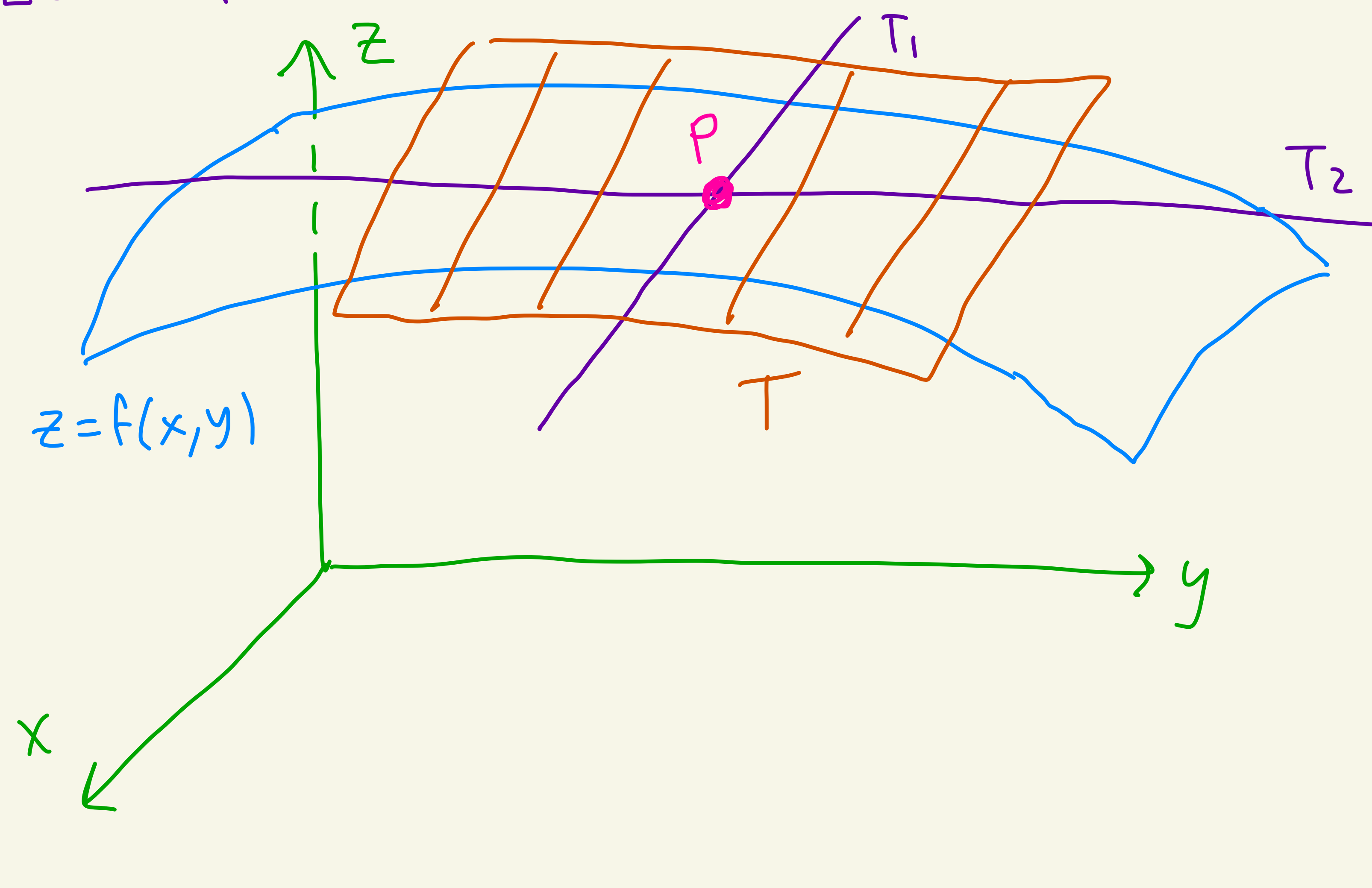
$$\frac{a}{c}(x-x_0)+\frac{b}{c}(y-y_0)+(z-z_0)=0$$

which gives

$$\boxed{z-z_0 = A(x-x_0)+B(y-y_0)} \quad (*)$$

Where $A=-\frac{a}{c}$, $B=-\frac{b}{c}$.

Let's find a formula for A and B.



Plugging in $y = y_0$ into $(*)$ gives

$$z - z_0 = A(x - x_0) + \underbrace{B(y_0 - y_0)}_{0}$$

which is

$$z - z_0 = A(x - x_0)$$

This is the equation for T_1

which has slope $A = f_x(x_0, y_0)$.

Plugging in $x = x_0$ into $(*)$ gives

$$z - z_0 = \underbrace{A(x_0 - x_0)}_{0} + B(y - y_0)$$

which is

$$z - z_0 = B(y - y_0)$$

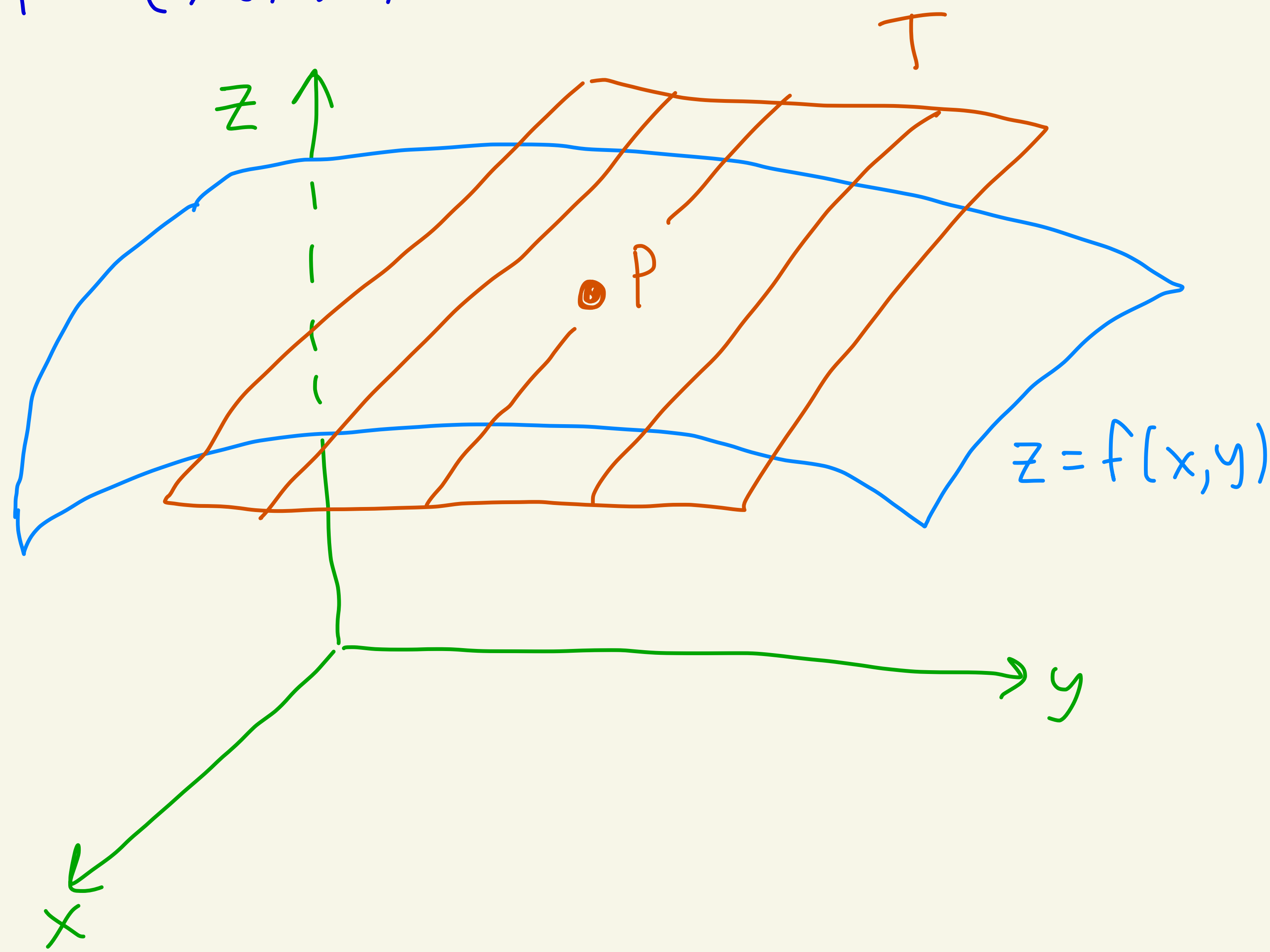
This is the equation for T_2

which has slope $B = f_y(x_0, y_0)$

Thus, T has equation

$$\boxed{z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

We call T the <u>tangent plane</u> to the surface S at the point $P=(x_0,y_0,z_0)$.



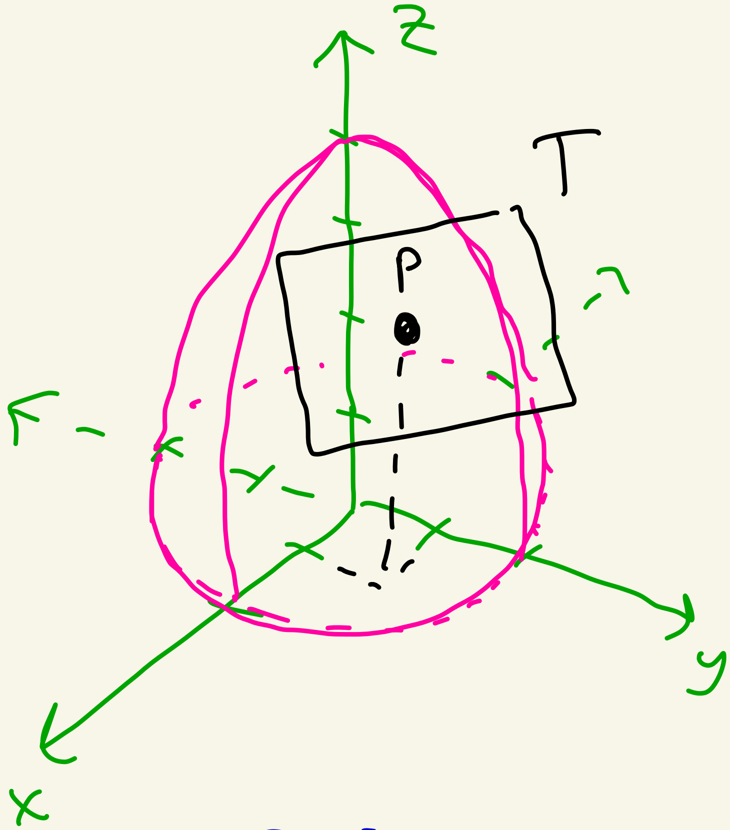
Ex: Consider the surface S given by $z = f(x,y)$ where $f(x,y) = 4 - x^2 - y^2$.

We have that

$$f_x = -2x \quad \text{and} \quad f_y = -2y$$

Let's find the tangent plane at $P = (1,1,2)$.

$z = 4 - 1^2 - 1^2 = 2$
at $x=1, y=1$



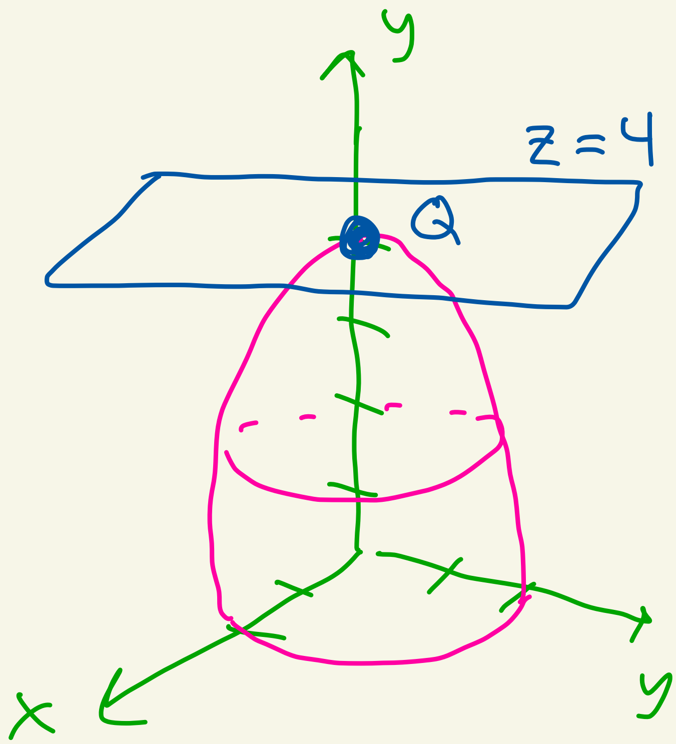
We have

$$f_x(1,1) = -2$$
$$f_y(1,1) = -2$$

So the tangent plane at P is

$$z - 2 = -2(x-1) - 2(y-1)$$

What is the tangent plane at $Q = (0,0,4)$?



There we have

$f_x(0,0) = 0$

$f_y(0,0) = 0$

So the tangent plane is

$$z - 4 = 0(x-0) + 0(y-0)$$

which is

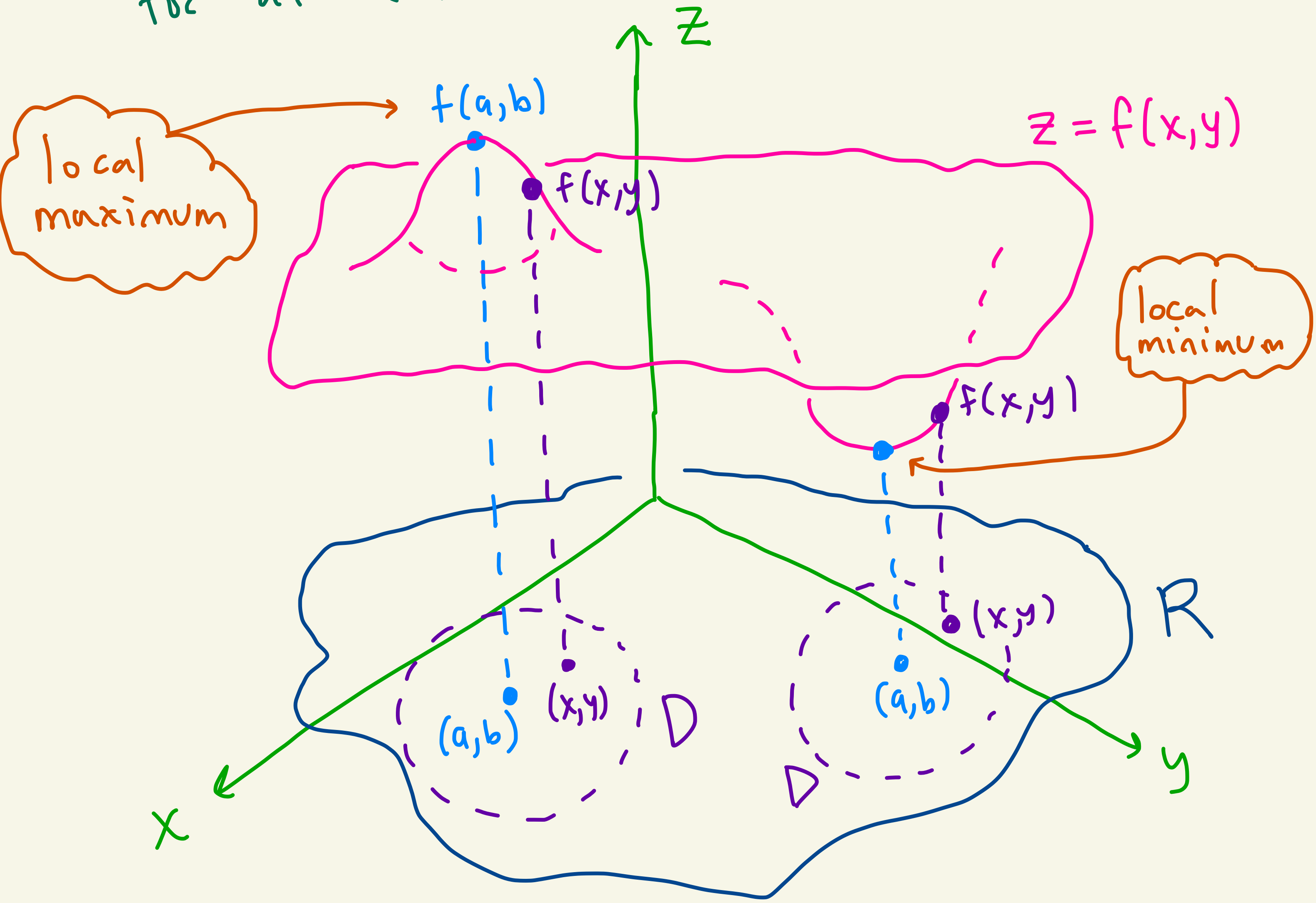
$$z = 4.$$

This is a flat horizontal plane which reflects the fact that $z = 4 - x^2 - y^2$ has a maximum value at Q.

We now use the tangent plane to find where the local maximums and minimums of a surface are

<u>Def:</u> Let $f(x,y)$ be defined on R.

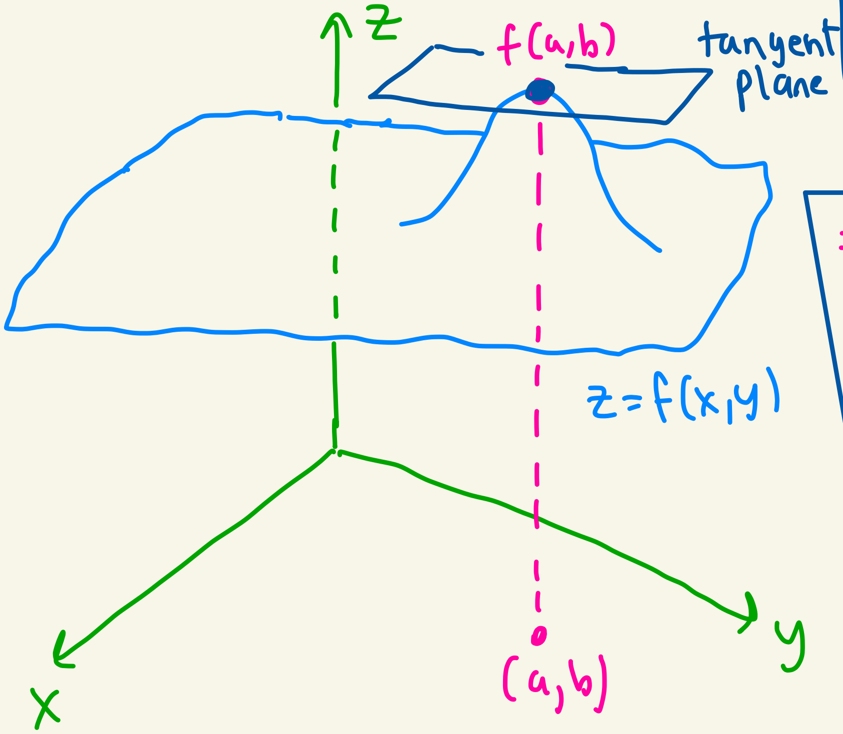
- We say that (a,b) is a <u>local maximum</u> of f if there exists an open disc D contained in R where $f(x,y) \le f(a,b)$ for all (x,y).



- We say that (a,b) is a <u>local minimum</u> of f if there exists an open disc D contained in R where $f(a,b) \leq f(x,y)$ for all (x,y).

<u>Theorem:</u> If $f(x,y)$ has a local maximum or minimum at (a,b) and f_x, f_y exist at (a,b), then $f_x(a,b)=0$ and $f_y(a,b)=0$.

picture for theorem



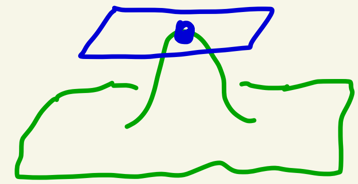
The local max/min occur when the tangent plane is horizontal. Tangent plane:

$$z - f(a,b) = \underbrace{f_x(a,b)}_{0}(x-a) + \underbrace{f_y(a,b)}_{0}(y-b)$$

which is horizontal when $f_x(a,b)=0$ and $f_y(a,b)=0$ giving $z=f(a,b)$ as the tangent plane.

Def: If either

① $f_x(a,b) = 0$ and $f_y(a,b) = 0$

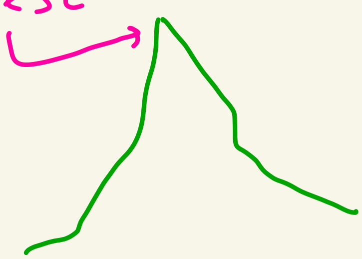


tangent plane is horizontal

or

② One of $f_x(a,b)$ or $f_y(a,b)$ does not exist

picks up cases like these



Then we call (a,b) a critical point of f.

Note: The local min/max's occur at critical points.

But a critical point might not be a local min or max

ex is saddle points below

Second Derivative Test

Suppose the second order partial derivatives of $f(x,y)$ are continuous on a disc centered at (a,b).

Suppose that

$$f_x(a,b)=0 \quad \text{and} \quad f_y(a,b)=0$$

(a,b) is a critical point

Let

$$D = f_{xx}(a,b)\cdot f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Then:

<u>case 1:</u> If $D>0$ and $f_{xx}(a,b)>0$ then (a,b) is a local minimum.

<u>case 2:</u> If $D>0$ and $f_{xx}(a,b)<0$ then (a,b) is a local maximum.

<u>case 3:</u> If $D<0$, then (a,b) is not a local minimum and is also not a local maximum.

these are called <u>saddle points</u>

Notes:

- If $D = 0$, then the test cannot be used. So no info is given.

- Can $D > 0$ and $f_{xx}(a,b) = 0$ happen?
  If so then $\underbrace{D}_{\text{positive}} = \underbrace{-\left[f_{xy}(a,b)\right]^2}_{\text{negative}}$

  So this case can't occur.

- One way to remember D is:

$$D = \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}}_{\text{determinant}} = f_{xx}f_{yy} - f_{xy}f_{yx}$$

$$\underset{\uparrow \atop f_{xy} = f_{yx}}{=} f_{xx}f_{yy} - \left(f_{xy}\right)^2$$

Ex: $f(x,y) = 4 - x^2 - y^2$

critical points:

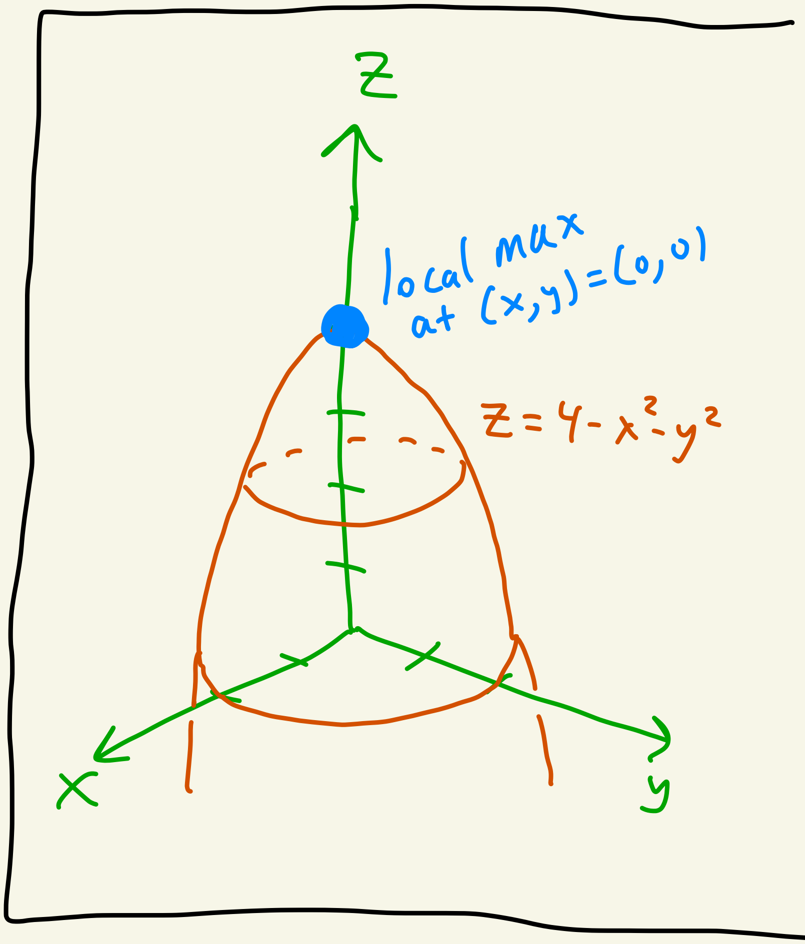
$$\boxed{\begin{aligned} f_x &= -2x \\ f_y &= -2y \end{aligned}}$$

$\downarrow$

$$\boxed{\begin{aligned} -2x &= 0 \\ -2y &= 0 \end{aligned}} \leftarrow \boxed{x = 0 \text{ and } y = 0}$$

$\downarrow$

$$\boxed{(x,y) = (0,0)} \leftarrow \boxed{\text{one critical point}}$$



2nd derivative test:

Let's test what happens at $(0,0)$

$f_{xx} = -2 \qquad f_{xy} = 0$

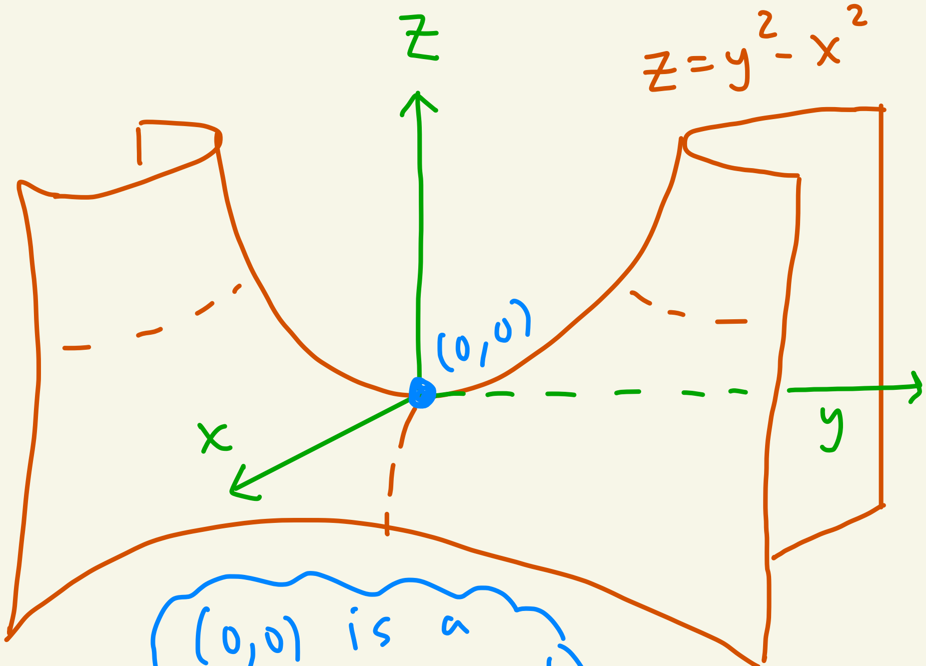
$f_{yy} = -2$

$D = (-2)(-2) - 0^2 = 4 > 0$

Since $D > 0$ and $f_{xx}(0,0) = -2 < 0$

we have that $\boxed{(0,0) \text{ is a local maximum.}}$

Ex: Let $f(x,y) = y^2 - x^2$.



(0,0) is a saddle point

critical points:

$$\boxed{\begin{aligned} f_x &= -2x \\ f_y &= 2y \end{aligned}}$$

$\downarrow$

$$\boxed{\begin{aligned} -2x &= 0 \\ 2y &= 0 \end{aligned}} \leftarrow \boxed{\begin{aligned} &x = 0 \\ &\text{and} \\ &y = 0 \end{aligned}}$$

$\downarrow$

$$\boxed{(x,y) = (0,0)}$$

2nd derivative test:

Let's see what kind of critical point $(0,0)$ is.

$f_{xx} = -2 \qquad f_{xy} = 0$

$f_{yy} = 2 \qquad f_{yy} = 0$

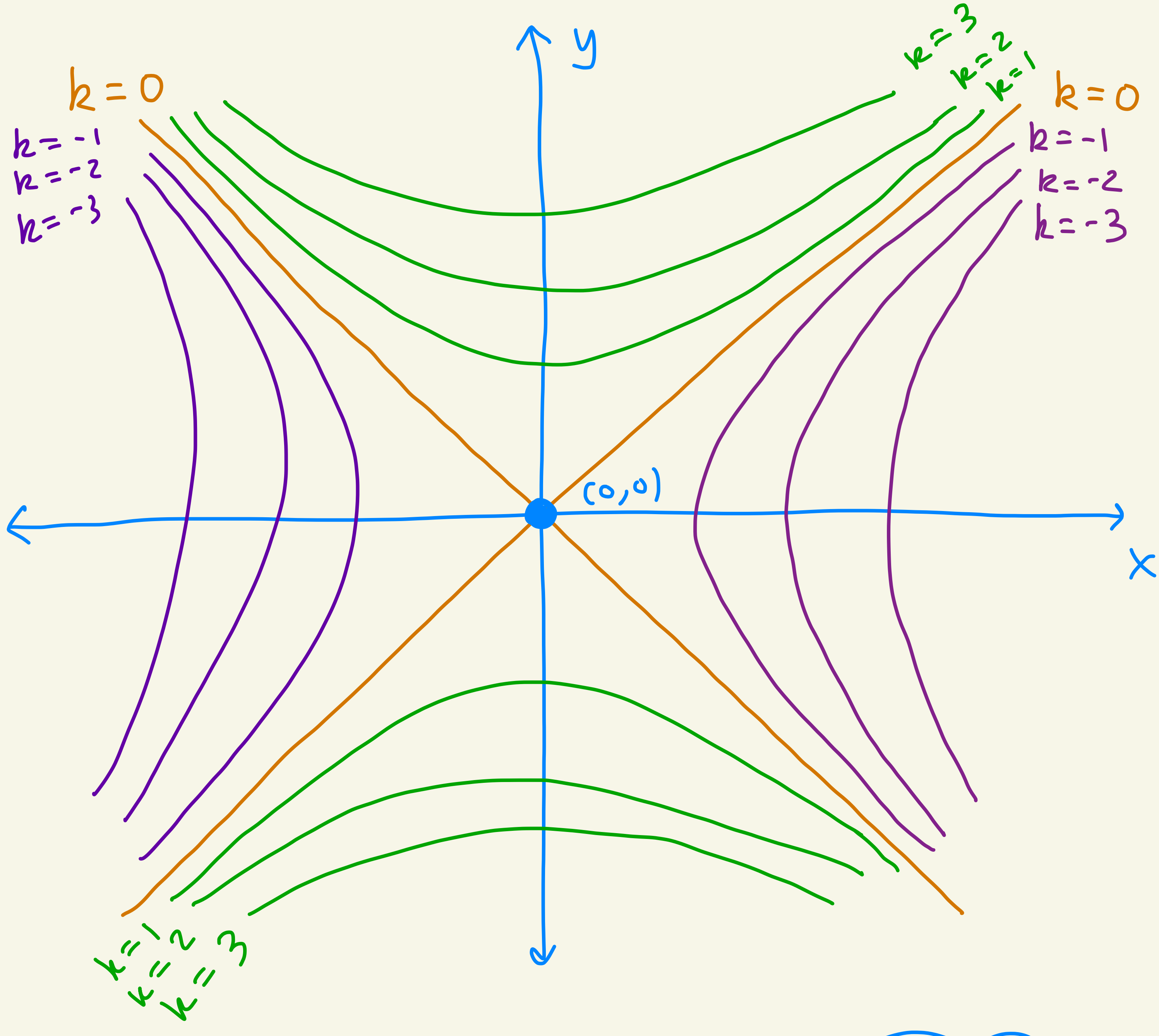
$D = f_{xx} f_{yy} - \left(f_{xy}^2\right) = -4$

Note that $D < 0$ always.

Case 3 of the 2nd derivative test applies.

$\boxed{(0,0) \text{ is a saddle point}} \leftarrow \boxed{\begin{aligned} &\text{not a local min} \\ &\text{not a local max} \end{aligned}}$

Some level curves of $z = y^2 - x^2$ are:



(0,0) is a saddle point

Ex: Let $f(x,y) = x^4 + y^4 - 4xy + 1$

Let's find the local minimums and maximums of f.

①

②

critical points:

$$\boxed{\begin{aligned} f_x &= 4x^3 - 4y \\ f_y &= 4y^3 - 4x \end{aligned}} \longrightarrow \boxed{\begin{aligned} 4x^3 - 4y &= 0 \\ 4y^3 - 4x &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} x^3 - y &= 0 \quad ① \\ y^3 - x &= 0 \quad ② \end{aligned}}$$

We want the (x,y) that solve both ① and ②.

① gives $y = x^3$.

Plug $y = x^3$ into ② to get:

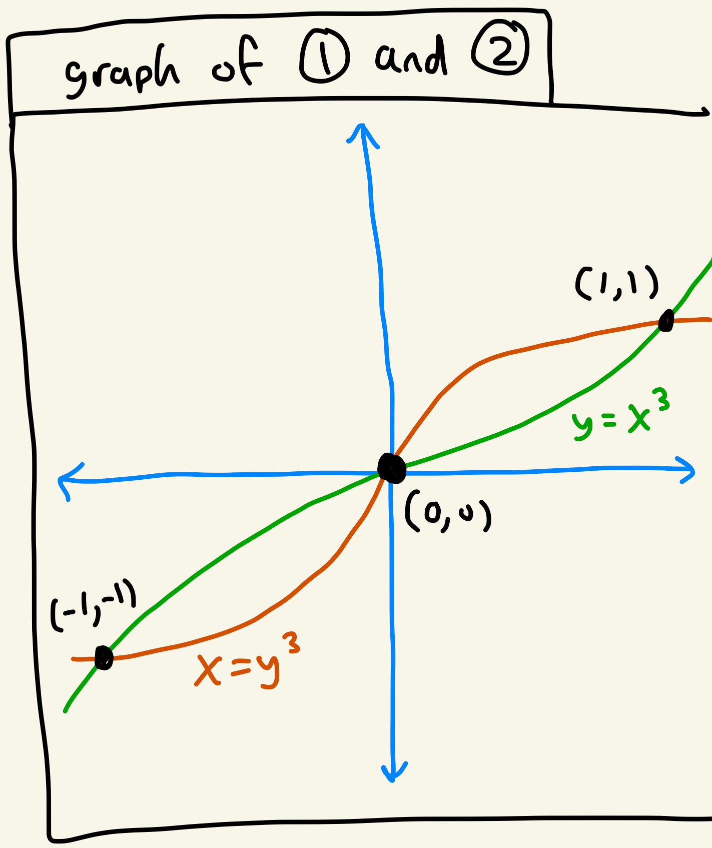
$$y^3 - x = 0$$
$$(x^3)^3 - x = 0$$
$$x^9 - x = 0$$
$$x(x^8 - 1) = 0$$
$$x(x^4 - 1)(x^4 + 1) = 0$$
$$x\underbrace{(x^2 - 1)}\underbrace{(x^2 + 1)}(x^4 + 1) = 0$$

$\boxed{x = 0}$

$\boxed{\begin{aligned} x^2 - 1 &= 0 \\ x &= \pm 1 \end{aligned}}$

$x^2 + 1 = 0$ can't happen

$x^4 + 1 = 0$ can't happen



So, $x = 0, 1, -1$.

Plug these back into $y = x^3$ from ① to get:

$$x = 0 \longrightarrow y = 0^3 = 0$$
$$x = 1 \longrightarrow y = 1^3 = 1$$
$$x = -1 \longrightarrow y = (-1)^3 = -1$$

The critical points are:

$$\boxed{(x,y) = (0,0),\ (1,-1),\ (-1,-1)}$$

<u>2nd derivative test</u>:

$f_x = 4x^3 - 4y \qquad f_{xx} = 12x^2 \qquad f_{xy} = -4$

$f_y = 4y^3 - 4x \qquad f_{yy} = 12y^2 \qquad f_{yx} = -4$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (12x^2)(12y^2) - (-4)^2$$
$$= 144x^2y^2 - 16$$

$\boxed{D(0,0) = 144 \cdot 0^2 \cdot 0^2 - 16 = -16 < 0}$ ← (0,0) is a saddle point

$\boxed{\begin{array}{l} D(1,1) = 144 \cdot 1^2 \cdot 1^2 - 16 = 128 > 0 \\ f_{xx}(1,1) = 12(1)^2 = 12 > 0 \end{array}}$ ← (1,1) is a local min

$\boxed{\begin{array}{l} D(-1,-1) = 144 \cdot (-1)^2 \cdot (-1)^2 - 16 = 128 > 0 \\ f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0 \end{array}}$ ← (-1,-1) is a local min

Thus the answer is:

Saddle points	local mins	local maxs
$(0,0)$	$(1,1), (-1,-1)$	none

Ex: If time do an optimization of distance problem here as an example.